Int J Theor Phys (2008) 47: 2297–2313
DOI 10.1007/s10773-008-9662-9

Unified Split Octonion Formulation of Dyons

P.S. Bisht · Shalini Dangwal · O.P.S. Negi

Received: 14 November 2007 / Accepted: 15 January 2008 / Published online: 2 February 2008

Abstract Demonstrating the split octonion formalism for unified fields of dyons (electromagnetic fields) and gravito-dyons (gravito-Heavisidian fields of linear gravity), relevant field equations are derived in compact, simpler and manifestly covariant forms. It has been shown that this unified model reproduces the dynamics of structure of fields associated with individual charges (masses) in the absence of others.

Keywords Split-octonion · Monopoles · Dyons · Gravitodyons · Gravito-Heavisidian fields

1 Introduction

Magnetic monopoles [1, 2] were advocated to symmetrize Maxwell's equations in a manifest way that the mere existence of an isolated magnetic charge implies the quantization of electric charge and accordingly the considerable literature [3–15] has come in force. The fresh interests are enhanced with the idea of Hooft [16] and Polyakov [17] that the classical solutions having the properties of magnetic monopoles may be found in Yang-Mills gauge theories. Julia and Zee [18] extended it to construct the theory of non Abelian dyons (particles [3–9] carrying simultaneously electric and magnetic charges). In view of the explanation of CP-violation in terms of non-zero vacuum angle of world [19], the monopoles are necessary dyons and Dirac quantization condition permits dyons to have analogous elec-

P.S. Bisht · S. Dangwal · O.P.S. Negi
Department of Physics, Kumaun University, S.S.J. Campus, Almora, 263601 Uttarakhand, India

P.S. Bisht
e-mail: ps_bisht123@rediffmail.com

S. Dangwal
e-mail: shalini_dangwal@rediffmail.com

O.P.S. Negi (✉)
Institute of Theoretical Physics, Chinese Academy of Sciences, KITP Building Room No. 6304, Hai Dian Qu Zhong Guan Chun Dong Lu 55 Hao, Beijing 100080, People's Republic of China
e-mail: ops_negi@yahoo.co.in

tric charge. The quantum mechanical excitation of fundamental monopoles include dyons which are automatically arisen [12, 13, 15] from the semi-classical quantization of global charge rotation degree of freedom of monopoles. Accordingly, the self-consistent and manifestly covariant theory of generalized electromagnetic fields associated with dyons (particles carrying electric and magnetic charges) has been discussed by us [20, 21].

The close analogy between Newton's gravitation law and Coulomb's law of electricity led many authors to investigate further similarities, such as the possibility that the motion of mass-charge could generate the analogous of a magnetic field which is produced by the motion of electric-charge, i.e. the electric current. So, there should be the mass current which would produce a magnetic type field namely 'gravitomagnetic' field. Maxwell [22] in one of his fundamental works on electromagnetism, turned his attention to the possibility of formulating the theory of gravitation similar to the electromagnetic equations. In 1893 Heaviside [23, 24] investigated the analogy between gravitation and electromagnetism where he explained the propagation of energy in a gravitational field, in terms of a gravito electromagnetic Poynting vector, even though he (just as Maxwell did) considered the nature of gravitational energy a mystery. The analogy has also been explored by Einstein [25], in the framework of General Relativity, and then by Thirring [26–28] and Lense and Thirring [29], that a rotating mass generates a gravito magnetic field causing a precession of planetary orbits. Expounding the basics of the gravito electromagnetic form of the Einstein equations,theory of gravito magnetism has also been reviewed by Ruggiero-Tartaglia [30].

As such, this analogy describes a structural symmetry between linear gravitational and usual electromagnetic fields and leads the asymmetry in Einstein's linear equation of gravity and accordingly suggests the existence [31, 32] of gravitational analogue of magnetic monopole. Like magnetic field, Cantani [33] introduced a new field (i.e. namely the Heavisidian field) depending upon the velocities of gravitational charges (masses) and derived the covariant equations (Maxwell's equations) of linear gravitational fields. Avoiding the use of arbitrary string variables [1, 2], we [34–36] have also formulated manifestly covariant theory of gravito-dyons in terms of two four-potentials and maintained the structural symmetry between generalized electromagnetic fields of dyons and generalized gravito-Heavisidian fields of gravito-dyons.

There has been a revival in the formulation of natural laws so that there exists [37] four-division algebras consisting the algebra of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and Octonions ($\mathcal{O}$). All four algebra's are alternative with totally anti symmetric associators. Quaternions [38, 39] were very first example of hyper complex numbers have been widely used [40–48] to the various applications of mathematics and physics. Since octonions share with complex numbers and quaternions, many attractive mathematical properties, one might except that they would be equally as useful as others. Octonion [49, 50] analysis has been widely discussed by Baez [51]. It has now played an important role in the context of various physical problems [52–63] of higher dimensional supersymmetry, super gravity and super strings etc. In recent years, it has also drawn interests of many [64–72] towards the developments of wave equation and octonion form of Maxwell's equations. We have also studied [73–76] octonion electrodynamics, dyonic field equation and octonion gauge analyticity of dyons consistently and obtained the corresponding field equations (Maxwell's equations) and equation of motion in compact and simpler formulation. Keeping these applications of octonions in mind, in the present paper, we have applied the formalism of split octonions to develop an unified model for generalized electromagnetic fields of dyons and those for generalized Gravito-Heavisidian fields of gravito dyons with the simultaneous existence of electric, magnetic, gravitational and Heavisidian charges (masses). We have thus obtained manifestly covariant forms of relevant field equations and

Table 1 Octonion multiplication table

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

derived the corresponding quantization parameters in consistent, compact, simpler forms. It has been shown that this unified theory reproduces the dynamics of individual charges (masses) in the absence of others.

2 Octonion Definition

An octonion x is expressed as a set of eight real numbers

$$\begin{aligned} x = (x_0, x_1, \ldots, x_7) &= x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 \\ &= x_0e_0 + \sum_{A=1}^{7} x_A e_A \quad (A = 1, 2, \ldots, 7), \end{aligned} \tag{1}$$

where e_A $(A = 1, 2, \ldots, 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. Set of octets $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ are known as the octonion basis elements and thus satisfy the following multiplication rules

$$\begin{gathered} e_0 = 1, \qquad e_0e_A = e_Ae_0 = e_A \quad (A = 1, 2, \ldots, 7), \\ e_Ae_B = -\delta_{AB}e_0 + f_{ABC}e_C \quad (A, B, C = 1, 2, \ldots, 7). \end{gathered} \tag{2}$$

The structure constants f_{ABC} are described as completely antisymmetric and take the value 1 for following combinations [51–72];

$$f_{ABC} = +1 = (123), (471), (257), (165), (624), (543), (736). \tag{3}$$

It is to be noted that the summation convention is used for repeated indices. Here the octonion algebra $\mathcal{O}$ is described over the algebra of rational numbers having the vector space of dimension 8. Octonion algebra is non associative and multiplication rules for its basis elements given by (2, 3) are then generalized in Table 1.

Hence we have

$$e_A(e_Be_C) \neq (e_Ae_B)e_C \tag{4}$$

and the commutation rules for octonion basis elements are given by

$$\begin{aligned} [e_A, e_B] &= 2f_{ABC}e_C; \\ \{e_A, e_B\} &= -\delta_{AB}e_0; \end{aligned} \tag{5}$$

where brackets [] and {} are used respectively for commutation and the anti commutation relations while δ_{AB} is the usual Kroneckar delta-Dirac symbol. Octonion conjugate is thus defined as,

$$\begin{aligned}\bar{x} &= x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 - x_5e_5 - x_6e_6 - x_7e_7,\\ &= x_0e_0 - \sum_{A=1}^{7} x_Ae_A \quad (A = 1, 2, \dots, 7),\end{aligned} \tag{6}$$

where we have used the conjugates of basis elements as

$$\overline{e_0} = e_0; \qquad \overline{e_A} = -e_A. \tag{7}$$

An Octonion can be decomposed in terms of its scalar ($Sc(x)$) and vector ($Vec(x)$) parts as

$$\begin{aligned}Sc(x) &= \frac{1}{2}(x + \bar{x}) = x_0,\\ Vec(x) &= \frac{1}{2}(x - \bar{x}) = \sum_{A=1}^{7} x_Ae_A.\end{aligned} \tag{8}$$

Conjugates of product of two octonions and its own are described as

$$\overline{(xy)} = \overline{y}\,\overline{x}; \qquad \overline{(\bar{x})} = x \tag{9}$$

while the scalar product of two octonions is defined as

$$\langle x, y\rangle = \sum_{\alpha=0}^{7} x_\alpha y_\alpha = \frac{1}{2}(x\bar{y} + y\bar{x}) = \frac{1}{2}(\bar{x}y + \bar{y}x) \tag{10}$$

which can be written in terms of octonion units as

$$\langle e_A, e_B\rangle = \frac{1}{2}(e_A\overline{e_B} + e_B\overline{e_A}) = \frac{1}{2}(\overline{e_A}e_B + \overline{e_B}e_A) = \delta_{AB}. \tag{11}$$

Following Catto [52] let us define

$$e_{AB} = \frac{1}{2}(\overline{e_A}e_B - \overline{e_B}e_A) \tag{12}$$

and

$$e'_{AB} = \frac{1}{2}(e_A\overline{e_B} - e_B\overline{e_A}). \tag{13}$$

Hence we may write

$$\overline{e_A}e_B = \frac{1}{2}(\overline{e_A}e_B + \overline{e_B}e_A) + \frac{1}{2}(\overline{e_A}e_B - \overline{e_B}e_A) = \delta_{AB} + e_{AB} \tag{14}$$

and

$$e_A\overline{e_B} = \frac{1}{2}(e_A\overline{e_B} + e_B\overline{e_A}) + \frac{1}{2}(e_A\overline{e_B} - e_B\overline{e_A}) = \delta_{AB} + e'_{AB}. \tag{15}$$

Equations (12) and (13) may be interpreted as the dyadic anti symmetric tensors and can be written component wise as

$$e_{AB} = e'_{AB} = -f_{ABC}e_C; \qquad e_{0A} = e'_{0A} = e_A. \tag{16}$$

It shows that octonions describe the covariant formulations in eight dimensional space. The norm of the octonion $N(x)$ is defined as

$$N(x) = \overline{x}x = x\bar{x} = \sum_{\alpha=0}^{7} x_\alpha^2 e_0 \tag{17}$$

which is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) = N(x)N(y) = N(y)N(x). \tag{18}$$

As such, for a nonzero octonion x, we define its inverse as

$$x^{-1} = \frac{\bar{x}}{N(x)} \tag{19}$$

which shows that

$$\begin{aligned} x^{-1}x = xx^{-1} &= 1.e_0, \\ (xy)^{-1} &= y^{-1}x^{-1}. \end{aligned} \tag{20}$$

Equation (4) shows that octonions are not associative and thus do not form the group in their usual form. Non-associativity of octonion algebra $\mathcal{O}$ is provided by the associator [52–86] defined for any 3 octonions as follows,

$$(x, y, z) = (xy)z - x(yz), \quad \forall x, y, z \in \mathcal{O} \tag{21}$$

which gives rise to the associator for octonion units as

$$(e_A, e_B, e_C) = 2\varepsilon_{ABCD}e_D, \quad \forall A, B, C, D = 1, 2, \ldots, 7. \tag{22}$$

Here ε_{ABCD} are totally antisymmetric and equal to unity for the following 7 combinations,

$$1247, \quad 1265, \quad 2345, \quad 2376, \quad 3146, \quad 3157 \quad \text{and} \quad 4576. \tag{23}$$

On the other hand, the quaternion algebra $\mathbb{H}$ satisfies the associativity and forms a group under multiplication. It is described as the sub algebra of octonions and thus can be represented in terms of unit matrix $\hat{1}$ and Pauli matrices σ_j as

$$e_0 \to \sigma_0 = \hat{1} \quad \text{and} \quad e_j \to -i\sigma_j \quad (\forall\, j = 1, 2, 3)\ (i = \sqrt{-1}). \tag{24}$$

It is trivial to check that the above map is an isomorphism i.e.

$$e_j e_k \quad \Rightarrow \quad -\sigma_j\sigma_k = -(\delta_{jk} + i\varepsilon_{jkl}\sigma_l) \quad \Leftrightarrow \quad -(\delta_{jk} + \varepsilon_{jkl}e_l) \quad (\forall j, k, l = 1, 2, 3). \tag{25}$$

As such, in contrast to $\mathbb{H}$, the Cayley algebra $\mathcal{O}$ cannot be represented by matrices with the usual multiplication rules due to its non associative nature. However, it is possible to represent octonions by matrices, provided one defines a special multiplication rule among them in terms of its split octonion basis elements.

Table 2 Split Octonion multiplication table

$\cdot$	$u_0^\star$	$u_1^\star$	$u_2^\star$	$u_3^\star$	u_0	u_1	u_2	u_3
$u_0^\star$	$u_0^\star$	$u_1^\star$	$u_2^\star$	$u_3^\star$	0	0	0	0
$u_1^\star$	0	0	u_3	$-u_2$	$u_1^\star$	$-u_0^\star$	0	0
$u_2^\star$	0	$-u_3$	0	u_1	$u_2^\star$	0	$-u_0^\star$	0
$u_3^\star$	0	u_2	$-u_1$	0	$u_3^\star$	0	0	$-u_0^\star$
u_0	0	0	0	0	u_0	u_1	u_2	u_3
u_1	u_1	$-u_0$	0	0	0	0	$u_3^\star$	$-u_2^\star$
u_2	u_2	0	$-u_0$	0	0	$-u_3^\star$	0	$u_1^\star$
u_3	u_3	0	0	$-u_0$	0	$u_2^\star$	$-u_1^\star$	0

3 Split Octonions

The split octonions are a non associative extension of quaternions (or the split quaternions). They differ from the octonion in the signature of quadratic form. The split octonions have a signature $(4,4)$ whereas the octonions have positive signature $(8,0)$. The Cayley algebra of octonions over the field of complex numbers $\mathbb{C}_{\mathbb{C}} = \mathbb{C} \otimes C$ is visualized as the algebra of split octonions with its following basis elements,

$$
\begin{aligned}
u_0 &= \frac{1}{2}(1 + ie_7), & u_0^\star &= \frac{1}{2}(1 - ie_7),\\
u_1 &= \frac{1}{2}(e_1 + ie_4), & u_1^\star &= \frac{1}{2}(e_1 - ie_4),\\
u_2 &= \frac{1}{2}(e_2 + ie_5), & u_2^\star &= \frac{1}{2}(e_2 - ie_5),\\
u_3 &= \frac{1}{2}(e_3 + ie_6), & u_3^\star &= \frac{1}{2}(e_3 - ie_6),
\end{aligned}
\tag{26}
$$

where $(i = \sqrt{-1})$ is usual complex imaginary number and commutes with all the seven octonion imaginary units e_A $(A = 1, 2, \dots, 7)$. Using the multiplication table of octonion we get the following multiplication table for split octonion basis elements u_β and $u_\beta^\star$ $(\beta = 0, 1, 2, 3)$ (see Table 2).

As such an octonion Z can be expressed in terms of split octonion basis elements as

$$
Z = z_o u_0 + z_o^\star u_0^\star + z_j u_j + z_j^\star u_j^\star, \tag{27}
$$

where z_β $(\beta = 0, 1, 2, 3)$ are the complex numbers and

$$
u_0 Z = z_\beta u_\beta = z; \qquad u_0^\star Z = z_\beta^\star u_\beta^\star = z^\star. \tag{28}
$$

Like octonions, split octonions are non commutative and non associative. They also form a composition algebra and satisfy (18). Split octonions also satisfy the Moufang identities and thus form the alternative algebra. Therefore, by Artin's theorem, the sub algebra generated by any two elements is associative and the set of all invertible elements (i.e. those elements for which $N(x) \neq 0$) describe a Moufang loop. We may now introduce a convenient realization for the split octonion basis elements $(u_0, u_0^\star, u_j, u_j^\star)$ $(j = 1, 2, 3)$ in terms of quaternion basis elements, $e_0 \to \sigma_0 = \hat{1}$ and $e_j \to -i\sigma_j$ as

$$\begin{aligned} u_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; & u_0^\star &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \\ u_j &= \begin{pmatrix} 0 & 0 \\ e_j & 0 \end{pmatrix}; & u_j^\star &= \begin{pmatrix} 0 & -e_j \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{29}$$

The split Cayley (octonion) algebra is thus expressed in terms of 2×2 Zorn's vector matrices components of which are scalar and vector parts of a quaternion i.e.

$$\mathcal{O} = \left\{ \begin{pmatrix} m & \vec{p} \\ \vec{q} & n \end{pmatrix} : m, n \in Sc(\mathbb{H});\ \vec{p}, \vec{q} \in Vec(\mathbb{H}) \right\}. \tag{30}$$

As such, we may also write an arbitrary split octonion A in terms of following 2×2 Zorn's vector matrix realizations as,

$$A = a u_0^\star + b u_0 + x_j u_j^\star + y_j u_j = \begin{pmatrix} a & -\vec{x} \\ \vec{y} & b \end{pmatrix}, \tag{31}$$

where a and b are scalars and $\vec{x}$ and $\vec{y}$ are three vectors. Thus the product of two octonions in terms of 2×2 Zorn's vector matrix realization is expressed as

$$\begin{pmatrix} a & \vec{x} \\ \vec{y} & b \end{pmatrix} \begin{pmatrix} c & \vec{u} \\ \vec{v} & d \end{pmatrix} = \begin{pmatrix} ac + \vec{x} \cdot \vec{v} & a\vec{u} + d\vec{x} - \vec{y} \times \vec{v} \\ c\vec{y} + b\vec{v} + \vec{x} \times \vec{u} & \vec{y} \cdot \vec{u} + bd \end{pmatrix}, \tag{32}$$

where $(\times)$ denotes the usual vector product, e_j $(j = 1, 2, 3)$ with $e_j \times e_k = \varepsilon_{jkl} e_l$ and $e_j e_k = -\delta_{jk}$. As such, we can relate the split octonions to the vector matrices given by (29). Octonion conjugate of (31) in terms of 2×2 Zorn's vector matrix realization is now defined as

$$\overline{A} = b u_0^\star + a u_0 - x_j u_j^\star - y_j u_j = \begin{pmatrix} b & \vec{x} \\ -\vec{y} & a \end{pmatrix}. \tag{33}$$

The norm of A is then defined as,

$$N(A) = A\overline{A} = \overline{A}A = (ab + \vec{x} \cdot \vec{y}) \cdot \hat{1} = n(A)\hat{1}, \tag{34}$$

where $\hat{1}$ is the identity element of the algebra given by $\hat{1} = 1u_0^\star + 1u_0$ and the expression $n(A) = (ab + \vec{x} \cdot \vec{y})$ defines the quadratic form which admits the composition $n(\vec{A}.\vec{B}) = n(\vec{A})n(\vec{B})$ for all $\vec{A}, \vec{B} \in \mathcal{O}$. As such, we may easily express the Euclidean or Minikowski four vector in split octonion formulation in terms of 2×2 Zorn's vector matrix realization. So, the space-time four-differential operator and its conjugates are then be written as

$$\begin{aligned} \boxdot &= \begin{pmatrix} \partial_4 & -\vec{\nabla} \\ \vec{\nabla} & \partial_4 \end{pmatrix} = \begin{pmatrix} i\partial_0 & -\vec{\nabla} \\ \vec{\nabla} & i\partial_0 \end{pmatrix}; \\ \overline{\boxdot} &= \begin{pmatrix} \partial_4 & \vec{\nabla} \\ -\vec{\nabla} & \partial_4 \end{pmatrix} = \begin{pmatrix} i\partial_0 & \vec{\nabla} \\ -\vec{\nabla} & i\partial_0 \end{pmatrix}. \end{aligned} \tag{35}$$

4 Duality Invariance and Generalized Fields of Dyons and Gravito-Dyons

Duality invariance is an old idea introduced a century ago in classical electromagnetism for Maxwell's equations in vacuum i.e.

$$\begin{aligned} \vec{\nabla}\cdot\vec{E}&=0; &\quad \vec{\nabla}\times\vec{E}&=-\frac{\partial\vec{M}}{\partial t},\\ \vec{\nabla}\cdot\vec{M}&=0; &\quad \vec{\nabla}\times\vec{M}&=\frac{\partial\vec{E}}{\partial t}, \end{aligned} \tag{36}$$

where $\vec{E}$ is the electric field and $\vec{M}$ is the magnetic fields. For brevity we have made use of the natural units ($c=\hbar=1$), and taking the other constants like gravitational constant as unity though out the text. Maxwell's equations in vacuum are symmetrical as well as invariant under both Lorentz transformations (in fact, conformal) and electromagnetic duality transformations given by,

$$\vec{E}\to\vec{E}\cos\theta+\vec{M}\sin\theta; \qquad \vec{M}\to-\vec{E}\sin\theta+\vec{M}\cos\theta. \tag{37}$$

For a particular value of $\theta=\frac{\pi}{2}$, (37) reduces to,

$$\begin{gathered} \vec{E}\to\vec{M}; \qquad \vec{M}\to-\vec{E} \quad \text{or}\\ \begin{pmatrix}\vec{E}\\ \vec{M}\end{pmatrix}\to\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{M}\end{pmatrix}. \end{gathered} \tag{38}$$

In terms of complex vectors the duality transformations are visualized as

$$(\vec{E}+i\vec{M})\to\exp(i\theta)(\vec{E}+i\vec{M}) \tag{39}$$

Lorentz invariance is obeyed even if we write Maxwell's equations in covariant formulation on introducing the electromagnetic field strengths. The duality symmetry is lost if electric charge and current source densities enter to Maxwell's equations. However, conventional Maxwell's equations are invariant under Lorentz and conformal transformations but neither these are symmetrical nor are invariant under the duality transformations (37, 38, 39). Dirac [1, 2] put forward this idea and introduced the concept of magnetic monopole not only to symmetrize the Maxwell's equations but also to make them dual invariant. Thus, electromagnetic duality requires:

- The existence of magnetic monopoles.
- The existence of magnetic monopole is closely related to the existence of a compact $U(1)$gauge group.
- The magnetic charge implies the C-invariance.
- Monopole equations are to be invariant under duality transformations.

Consequently, the Generalized Dirac Maxwell's (GDM) equations given below

$$\begin{aligned} \vec{\nabla}\cdot\vec{E}&=\rho_e; &\quad \vec{\nabla}\times\vec{E}&=-\frac{\partial\vec{M}}{\partial t}-\vec{j}_m;\\ \vec{\nabla}\cdot\vec{M}&=\rho_m; &\quad \nabla\times\vec{M}&=\vec{j}_e+\frac{\partial\vec{E}}{\partial t}; \end{aligned} \tag{40}$$

are invariant under duality transformations (37, 38, 39) incorporating the following duality among the electric and magnetic charge and current source densities

$$\begin{aligned}\begin{pmatrix}\rho_e\\ \rho_m\end{pmatrix} &\to \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\rho_e\\ \rho_m\end{pmatrix};\\ \begin{pmatrix}\vec{j}_e\\ \vec{J}_m\end{pmatrix} &\to \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\vec{j}_e\\ \vec{J}_m\end{pmatrix}.\end{aligned} \tag{41}$$

Here ρ_e is the electric charge source density, ρ_m is magnetic charge (monopole) source density, $\vec{j}_e$ is the electric current source density and $\vec{j}_m$ is magnetic current (monopole) source density.

Accordingly, on postulating the existence of Heavisidian monopole [31–34] and keeping in view the asymmetry therein between the gravitational (gravi-electric) and Heavisidian (gravi-magnetic) in Maxwellian gravity, the structural symmetry between these two interactions describes the invariance of GDM type equations for gravito-Heavisidian fields

$$\begin{aligned}\vec{\nabla}\cdot\vec{G} &= \rho_g; \qquad \vec{\nabla}\times\vec{G} = -\frac{\partial\vec{H}}{\partial t} - \vec{j}_h;\\ \vec{\nabla}\cdot\vec{H} &= \rho_h; \qquad \nabla\times\vec{H} = \vec{j}_h + \frac{\partial\vec{G}}{\partial t};\end{aligned} \tag{42}$$

under the following duality transformations,

$$\begin{aligned}\begin{pmatrix}\vec{G}\\ \vec{H}\end{pmatrix} &\to \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\vec{G}\\ \vec{H}\end{pmatrix};\\ \begin{pmatrix}\rho_g\\ \rho_h\end{pmatrix} &\to \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\rho_g\\ \rho_h\end{pmatrix};\\ \begin{pmatrix}\vec{j}_g\\ \vec{J}_h\end{pmatrix} &\to \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\vec{j}_g\\ \vec{J}_h\end{pmatrix}.\end{aligned} \tag{43}$$

Here $\vec{G}$ is the gravitational (gravi-electric) field , $\vec{H}$ is the Heavisidian (gravi-magnetic) fields, ρ_g is the gravitational (gravi-electric) charge (mass) density, ρ_m is Heavisidian (gravi-magnetic) monopole (mass) density, $\vec{j}_g$ is the gravitational (gravi-electric) current density and $\vec{j}_h$ is Heavisidian (gravi-magnetic) monopole current density in Maxwellian gravity (namely linear gravity). We may write these theories in covariant formulations accordingly by introducing the corresponding field strengths and gauge potentials. But the introduction of magnetic (Heavisidian) monopole (mass) leads to various discrepancies like the string singularity and even the Dirac quantization condition is no more dual invariant. Then the theories of dyons (particles carrying simultaneous existence of electric and magnetic charges) come in force and Dirac quantization condition have been made dual invariant by replacing it with Schwinger-Zwanzinger [3–9] quantization condition. In order to avoid the use of arbitrary string variables and keeping in mind the idea of two four-potentials, we have developed a manifestly covariant and dual invariant theory of generalized electromagnetic fields of dyons [20, 21] and accordingly those of generalized gravito-Heavisidian fields of gravito-dyons [34–36] on assuming the generalized charge, four-potential, vector field, current and generalized field tensors of dyons (gravito dyons) as a complex (order

Table 3

Dynamical variables	Fields associated with dyons	Fields associated with Gravito-dyons
Generalized charge (mass)	$Q^{EM}=(e,g)=(e+ig)$	$Q^{GH}=(m,h)=(m+ih)$
Generalized four-potential	$V_\mu^{EM}=(A_\mu,B_\mu)=(A_\mu+iB_\mu)$	$V_\mu^{GH}=(C_\mu,D_\mu)=(C_\mu+iD_\mu)$
Generalized four-current	$J_\mu^{EM}=(j_\mu^{(E)},j_\mu^{(M)})=(j_\mu^{(E)}+ij_\mu^{(M)})$	$J_\mu^{GH}=(j_\mu^{(G)},j_\mu^{(H)})=(j_\mu^{(G)}+ij_\mu^{(H)})$
Generalized vector-field	$\vec{\psi}^{EM}=(\vec{E},\vec{M})=(\vec{E}+i\vec{M})$	$\vec{\psi}^{GH}=(\vec{G},\vec{H})=(\vec{G}+i\vec{H})$
Generalized field tensor	$F_{\mu\nu}^{EM}=(A_{\mu\nu},B_{\mu\nu})=(A_{\mu\nu}+iB_{\mu\nu})$	$F_{\mu\nu}^{GH}=(C_{\mu\nu},D_{\mu\nu})=(C_{\mu\nu}+iD_{\mu\nu})$

pair of two real numbers) ones like (39) with their real and imaginary parts as a electric (gravitational) and magnetic (Heavisidian) constituents. Hence the Generalized Dirac Maxwell's (GDM) equations given by (40) and (42) are respectively the field equations of dyons and gravito-dyons. Let us summaries these two theories in Table 3, where e is electric charge, g is magnetic charge, m is the gravitational charge (mass), h is Heavisidian charge (mass), A_μ is electric four-potential, B_μ is magnetic four-potential, C_μ is gravitational four-potential, D_μ is Heavisidian four-potential, $j_\mu^{(E)}=\{\rho_e,-\vec{j}_e\}$ is electric four-current, $j_\mu^{(M)}=\{\rho_m,-\vec{j}_m\}$ is magnetic four-current, $j_\mu^{(G)}=\{\rho_g,-\vec{j}_g\}$ is gravitational four-current and $j_\mu^{(H)}=\{\rho_h,-\vec{j}_h\}$ is Heavisidian four-current. *EM* stands for electromagnetic, *GH* is used for gravito-Heavisidian; $A_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}=\partial_\nu A_\mu-\partial_\mu A_\nu$; $B_{\mu\nu}=B_{\mu,\nu}-B_{\nu,\mu}=\partial_\nu B_\mu-\partial_\mu B_\nu$; $C_{\mu\nu}=C_{\mu,\nu}-C_{\nu,\mu}=\partial_\nu C_\mu-\partial_\mu C_\nu$ and $D_{\mu\nu}=D_{\mu,\nu}-D_{\nu,\mu}=\partial_\nu D_\mu-\partial_\mu D_\nu$ $(\mu,\nu=0,1,2,3)$. Here the real parameters of complex variables are described as the electric (gravitational) constituents while their imaginary counter parts are identified as the magnetic (Heavisidian) constituents of dyons (gravito-dyons). These generalized fields and their quantum equations will be described in details in covariant formulation in the next sections with the applications of split octonions. As such, duality transformations for these dynamical variables associated with dyons (gravito-dyons) in generalized electromagnetic (gravito-Heavisidian) fields take the following forms,

$$\begin{aligned}
\text{Generalized charge (mass) } Q &\to (\exp i\theta).(Q),\\
\text{Generalized four-potential } V_\mu &\to (\exp i\theta).(V_\mu),\\
\text{Generalized four-current } J_\mu &\to (\exp i\theta).(J_\mu),\\
\text{Generalized vector-field } \vec{\psi} &\to (\exp i\theta).(\vec{\psi}),\\
\text{Generalized field tensor } F_{\mu\nu} &\to (\exp i\theta).(F_{\mu\nu}).
\end{aligned} \tag{44}$$

Hence with these transformations the GDM equations (40, 42), corresponding covariant field equations, equation of motion, Schwinger-Zwanzinger [3–9] quantization condition, BPS mass formula and the energy-momentum densities of generalized electromagnetic (gravito-Heavisidian) fields of dyons (gravito-dyons) in complex representation remain invariant. The duality conjecture has now been gaining enormous potential importance in connection with latest developments of elementary particles in gauge theories, grand unified theories, supersymmetry and super strings.

5 Split Octonion Formulation for Unified Fields of Dyons

We may now apply the split octonion formalism in order to formulate the unified theory of generalized electromagnetic fields of dyons [20, 21] and those of generalized gravito-

Heavisidian fields of gravito-dyons [34–36] discussed above. Let us combine the generalized charges of dyons and gravito-dyons (i.e. both complex quantities) with the help of Cayley Dickson process to make them an unified quaternion tetrad. As such we may express the quaternion charge for the unified fields of dyons and gravito-dyons as

$$Q=(e+ig)+(m+ih)j=e+ig+jm+ijh=e+ig+jm+kh=(e,g,m,h), \quad (45)$$

where i, j, k are the three non commutating quaternion imaginary elements $i^2=j^2=k^2=-1$; $ij=-ji=k$; $jk=-kj=i$; $ki=-ik=j$ and we may replace them by the quaternion units e_1, e_2 and e_3 of quaternion tetrad $(1,e_1,e_2,e_3)$ which satisfies the multiplication rules given by (25). Unfortunately, the quaternion basis elements loose the matrix realization when we write them in split basis. Like (31), we may now define the split octonion representation of unified quaternion charge of dyons and gravito-dyons in terms of 2×2 Zorn's vector matrix realization as

$$\begin{aligned} Q=(e,g,m,h)&=\begin{Bmatrix} e & -e_1g-e_2m-e_3h \\ e_1g+e_2m+e_3h & e \end{Bmatrix} \\ &=e(u_0^\star+u_0)+g(u_1+u_1^\star)+m(u_2+u_2^\star)+h(u_3+u_3^\star). \end{aligned} \quad (46)$$

Like (33), we may write split octonion conjugate of unified quaternion charge of dyons and gravito-dyons in terms of 2×2 Zorn's vector matrix realization as

$$\begin{aligned} \overline{Q}=(e,-g,-m,-h)&=\begin{Bmatrix} e & e_1g+e_2m+e_3h \\ -e_1g-e_2m-e_3h & e \end{Bmatrix} \\ &=e(u_0^\star+u_0)-g(u_1+u_1^\star)-m(u_2+u_2^\star)-h(u_3+u_3^\star). \end{aligned} \quad (47)$$

The norm of split octonion form of unified quaternion charge of dyons and gravito-dyons is defined as,

$$\begin{aligned} N(Q)&=Q\overline{Q}=\overline{Q}Q \\ &=\begin{pmatrix} e^2+g^2+m^2+h^2 & 0 \\ 0 & e^2+g^2+m^2+h^2 \end{pmatrix}.\hat{1}=(e^2+g^2+m^2+h^2).\hat{1}. \end{aligned} \quad (48)$$

The interaction of ath split octonionic charge Q_a in the field of other bth split octonionic charge Q_b now depends on the quantity,

$$\begin{aligned} Q_a.\overline{Q_b}&=u_0(e_ae_b+m_am_b+g_ag_b+h_ah_b)+u_1(-e_ag_b+g_ae_b+m_ah_b-h_am_b) \\ &\quad+u_2(-e_am_b+m_ae_b+h_ag_b-g_ah_b)+u_3(-e_ah_b+h_ae_b+g_am_b-m_ag_b) \\ &\quad+u_0^\star(e_ae_b+m_am_b+g_ag_b+h_ah_b)+u_1^\star(h_am_b-m_ah_b-e_ag_b+g_ae_b) \\ &\quad+u_2^\star(g_ah_b-h_ag_b-e_am_b+m_ae_b)+u_3^\star(m_ag_b-g_am_b-e_ah_b+h_ae_b) \end{aligned} \quad (49)$$

and

$$\begin{aligned} \overline{Q_a}.Q_b&=u_0(e_ae_b+m_am_b+g_ag_b+h_ah_b)+u_1(e_ag_b-g_ae_b+m_ah_b-h_am_b) \\ &\quad+u_2(e_am_b-m_ae_b+h_ag_b-g_ah_b)+u_3(e_ah_b-h_ae_b+g_am_b-m_ag_b) \\ &\quad+u_0^\star(e_ae_b+m_am_b+g_ag_b+h_ah_b)+u_1^\star(h_am_b-m_ah_b+e_ag_b-g_ae_b) \\ &\quad+u_2^\star(g_ah_b-h_ag_b+e_am_b-m_ae_b)+u_3^\star(m_ag_b-g_am_b+e_ah_b-h_ae_b). \end{aligned} \quad (50)$$

Like (31), we may now introduce the unified split octonion form of quaternion valued four-potential of dyons and gravito-dyons in terms of 2×2 Zorn's vector matrix realization as,

$$V = \begin{pmatrix} A_0 + B_0 + C_0 + D_0 & -(\vec{A} + \vec{B} + \vec{C} + \vec{D}) \\ (\vec{A} + \vec{B} + \vec{C} + \vec{D}) & A_0 + B_0 + C_0 + D_0 \end{pmatrix} = \begin{pmatrix} V_0 & -\vec{V} \\ \vec{V} & V_0 \end{pmatrix}, \tag{51}$$

where A, B, C and D are the quaternionic forms of four-potential associated with electric, magnetic, gravitational (g-electric) and Heavisidian (g-magnetic) charges respectively. These are also written as follows in split octonionic formulation in terms of 2×2 Zorn's vector matrix realization,

$$\begin{aligned} A &= \begin{pmatrix} A_0 & -\vec{A} \\ \vec{A} & A_0 \end{pmatrix} = \begin{pmatrix} A_0 & -(A_1 e_1 + A_2 e_2 + A_3 e_3) \\ (A_1 e_1 + A_2 e_2 + A_3 e_3) & A_0 \end{pmatrix}; \\ B &= \begin{pmatrix} B_0 & -\vec{B} \\ \vec{B} & B_0 \end{pmatrix} = \begin{pmatrix} B_0 & -(B_1 e_1 + B_2 e_2 + B_3 e_3) \\ (B_1 e_1 + B_2 e_2 + B_3 e_3) & B_0 \end{pmatrix}; \\ C &= \begin{pmatrix} C_0 & -\vec{C} \\ \vec{C} & C_0 \end{pmatrix} = \begin{pmatrix} C_0 & -(C_1 e_1 + C_2 e_2 + C_3 e_3) \\ (C_1 e_1 + C_2 e_2 + C_3 e_3) & C_0 \end{pmatrix}; \\ D &= \begin{pmatrix} D_0 & -\vec{D} \\ \vec{D} & D_0 \end{pmatrix} = \begin{pmatrix} D_0 & -(D_1 e_1 + D_2 e_2 + D_3 e_3) \\ (D_1 e_1 + D_2 e_2 + D_3 e_3) & D_0 \end{pmatrix}. \end{aligned} \tag{52}$$

As such, we have reformulated the four vector potentials of all the individual charges namely, electric, magnetic, gravitational (g-electric) and Heavisidian (g-magnetic), by the virtue of split octonion analyticity in terms of 2×2 Zorn's vector matrix realization. Accordingly, the split octonion form of quaternion unified vector field of dyons and gravito-dyons may then be expressed as

$$\vec{\Psi} = \begin{pmatrix} 0 & -(\vec{E} + \vec{M} + \vec{G} + \vec{H}) \\ (\vec{E} + \vec{M} + \vec{G} + \vec{H}) & 0 \end{pmatrix}, \tag{53}$$

where C, $\vec{M}$, $\vec{G}$ and $\vec{H}$ are respectively the generalized electric, magnetic, gravitational and Heavisidian fields described in terms of two four potential theory of dyons and gravito-dyons with the following definitions,

$$\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A_0 - \vec{\nabla} \times \vec{B}; \\ \vec{M} &= -\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} B_0 + \vec{\nabla} \times \vec{A}; \\ \vec{G} &= -\frac{\partial \vec{C}}{\partial t} - \vec{\nabla} C_0 + \vec{\nabla} \times \vec{D}; \\ \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{\nabla} D_0 + \vec{\nabla} \times \vec{C}; \end{aligned} \tag{54}$$

where $\vec{\nabla} = (\partial_1, \partial_2, \partial_3) = (e_1 \partial_1 + e_2 \partial_2, +e \partial_3)$ and these generalized electric, magnetic, gravitational and Heavisidian fields satisfy the pairs of Generalized Dirac-Maxwell's (GDM) equations (40, 42) of dyons and gravito-dyons. the components of split octonion valued unified four potential V given by (51) and unified field $\vec{\Psi}$ given by (53) establish the following relation among them as,

$$\vec{\Psi} = -\frac{\partial \vec{V}}{\partial t} - \vec{\nabla} V_0 + i \vec{\nabla} \times \vec{V}. \tag{55}$$

Operating $\boxdot$ given by (35) to (51) and using the properties of multiplication of split octonion algebra n terms of 2×2 Zorn's vector matrix realization, we get the following split octonion form of potential field equation for the unified theory of dyons and gravito-dyons as,

$$\boxdot V = \Psi, \tag{56}$$

where

$$\Psi = \begin{pmatrix} \partial_0 V_0 + \vec{\nabla} \cdot \vec{V} & \frac{\partial \vec{V}}{\partial t} + \vec{\nabla} V_0 - i \vec{\nabla} \times \vec{V} \\ -\frac{\partial \vec{V}}{\partial t} - \vec{\nabla} V_0 + i \vec{\nabla} \times \vec{V} & \partial_0 V_0 + \vec{\nabla} \cdot \vec{V} \end{pmatrix} = \begin{pmatrix} 0 & -\vec{\Psi} \\ \vec{\Psi} & 0 \end{pmatrix}$$

is the split octonion form of unified vector field while the diagonal components are vanishing due to the Lorentz gauge conditions applied to each four potentials. Equation (56) is the split octonion potential wave equation for the unified fields of dyons and those of gravito dyons. This equation may also be visualized as the analogue of unified GDM equations of dyons and gravito-dyons and thus remains invariant under duality, quaternion and Lorentz transformations. As such, the unified potential field equation (56) is simple, compact, consistent and manifestly covariant. Accordingly, we may write the split octonion representation for four-current associated with the unified fields of dyons and gravito-dyons, the components of which are given by (40, 42), in the following manner,

$$J = \begin{pmatrix} \rho_e + \rho_g + \rho_m + \rho_h & -(\vec{j}_e + \vec{j}_g + \vec{j}_m + \vec{j}_h) \\ (\vec{j}_e + \vec{j}_g + \vec{j}_m + \vec{j}_h) & \rho_e + \rho_g + \rho_m + \rho_h \end{pmatrix} = \begin{pmatrix} J_0 & -\vec{J} \\ \vec{J} & J_0 \end{pmatrix}, \tag{57}$$

where

$$\begin{aligned} \boxdot\overline{\boxdot} A &= \overline{\boxdot}\,\boxdot\, A = j_e = \begin{pmatrix} \rho_e & -\vec{j}_e \\ \vec{j}_e & \rho_e \end{pmatrix}; \\ \boxdot\overline{\boxdot} B &= \overline{\boxdot}\,\boxdot\, B = j_g = \begin{pmatrix} \rho_g & -\vec{j}_g \\ \vec{j}_g & \rho_g \end{pmatrix}; \\ \boxdot\overline{\boxdot} C &= \overline{\boxdot}\,\boxdot\, C = j_m = \begin{pmatrix} \rho_m & -\vec{j}_m \\ \vec{j}_m & \rho_m \end{pmatrix}; \\ \boxdot\overline{\boxdot} D &= \overline{\boxdot}\,\boxdot\, D = j_h = \begin{pmatrix} \rho_h & -\vec{j}_h \\ \vec{j}_h & \rho_h \end{pmatrix} \end{aligned} \tag{58}$$

and

$$\boxdot\overline{\boxdot} = \overline{\boxdot}\boxdot = (\partial_4^2 + \partial_1^2 + \partial_2^2 + \partial_3^2).\hat{1} = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).\hat{1}. \tag{59}$$

Hence we may write the split octionion form of unified GDM equations in the following manners as the connection between potential and current,

$$\boxdot\overline{\boxdot} V = \overline{\boxdot}\,\boxdot\, V = \begin{pmatrix} J_0 & -\vec{J} \\ \vec{J} & J_0 \end{pmatrix} = J. \tag{60}$$

Here we may also obtain the split octonionic forms of Lorentz gauge condition as well as the continuity equation. These are described in terms of the inner products of two split octonions i.e. the inner product of split octonion differential operator respectively with potential and current of unified fields of dyons and gravito-dyons. Similarly, Split octonion representation for field strength tensor $Q_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu} = F^{EM}_{\mu\nu} + F^{GH}_{\mu\nu} = (A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, D_{\mu\nu})$ for the unified fields of dyons and gravito-dyons may be described as,

$$\begin{aligned} \mathcal{F} &= \begin{pmatrix} \partial_\mu V_\mu & -e_j(\partial_0 V_j + \partial_j V_0 + i\varepsilon_{jkl}\partial_k V_l) \\ e_j(\partial_0 V_j + \partial_j V_0 + i\varepsilon_{jkl}\partial_k V_l) & \partial_\mu V_\mu \end{pmatrix} \\ &\Rightarrow \quad \begin{pmatrix} 0 & -e_j \Psi_j \\ e_j \Psi_j & 0 \end{pmatrix} \quad \Leftrightarrow \quad \Psi, \end{aligned} \tag{61}$$

where $(j,k,l = 1,2,3;\ \mu,\nu = 0,1,2,3)$ and $i = \sqrt{-1}$ while the field strengths $A_{\mu\nu}$, $B_{\mu\nu}$, $C_{\mu\nu}$, $D_{\mu\nu}$ are respectively associated with individual electric, magnetic, gravitational and Heavisidian charges (masses) reduce to the following split octonionic forms,

$$\begin{aligned} \mathcal{F}^e &= \begin{pmatrix} \partial_\mu A_\mu & -e_j(\partial_0 A_j + \partial_j A_0 + i\varepsilon_{jkl}\partial_k A_l) \\ e_j(\partial_0 A_j + \partial_j A_0 + i\varepsilon_{jkl}\partial_k A_l) & \partial_\mu A_\mu \end{pmatrix} \quad \Leftrightarrow \quad \Psi^e; \\ \mathcal{F}^g &= \begin{pmatrix} \partial_\mu B_\mu & -e_j(\partial_0 B_j + \partial_j B_0 + i\varepsilon_{jkl}\partial_k B_l) \\ e_j(\partial_0 B_j + \partial_j B_0 + i\varepsilon_{jkl}\partial_k B_l) & \partial_\mu B_\mu \end{pmatrix} \quad \Leftrightarrow \quad \Psi^g; \\ \mathcal{F}^m &= \begin{pmatrix} \partial_\mu C_\mu & -e_j(\partial_0 C_j + \partial_j C_0 + i\varepsilon_{jkl}\partial_k C_l) \\ e_j(\partial_0 C_j + \partial_j C_0 + i\varepsilon_{jkl}\partial_k C_l) & \partial_\mu C_\mu \end{pmatrix} \quad \Leftrightarrow \quad \Psi^m; \\ \mathcal{F}^h &= \begin{pmatrix} \partial_\mu D_\mu & -e_j(\partial_0 D_j + \partial_j D_0 + i\varepsilon_{jkl}\partial_k D_l) \\ e_j(\partial_0 D_j + \partial_j D_0 + i\varepsilon_{jkl}\partial_k D_l) & \partial_\mu D_\mu \end{pmatrix} \quad \Leftrightarrow \quad \Psi^h. \end{aligned} \tag{62}$$

Similarly, we may write the split octonion form of the generalized field strengths $F^{EM}_{\mu\nu} = V^{EM}_{\mu,\nu} - V^{EM}_{\nu,\mu}$ of generalized electromagnetic fields of dyons and those $F^{GH}_{\mu\nu} = V^{GH}_{\mu,\nu} - V^{GH}_{\nu,\mu}$ for the generalized gravito-Heavisidian fields of gravito-dyons in the following manner,

$$\begin{aligned} \mathcal{F}^{EM} &= \begin{pmatrix} \partial_\mu V^{EM}_\mu & -e_j(\partial_0 V^{EM}_j + \partial_j V^{EM}_0 + i\varepsilon_{jkl}\partial_k V^{EM}_l) \\ e_j(\partial_0 V^{EM}_j + \partial_j V^{EM}_0 + i\varepsilon_{jkl}\partial_k V^{EM}_l) & \partial_\mu V^{EM}_\mu \end{pmatrix} \\ &\Leftrightarrow \quad \Psi^{EM}; \\ \mathcal{F}^{GH} &= \begin{pmatrix} \partial_\mu V^{GH}_\mu & -e_j(\partial_0 V^{Gh}_j + \partial_j V^{GH}_0 + i\varepsilon_{jkl}\partial_k V^{GH}_l) \\ e_j(\partial_0 V^{GH}_j + \partial_j V^{GH}_0 + i\varepsilon_{jkl}\partial_k V^{GH}_l) & \partial_\mu V^{Gh}_\mu \end{pmatrix} \\ &\Leftrightarrow \quad \Psi^{GH}. \end{aligned} \tag{63}$$

Unified split octonion valued field tensor $Q_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu} = F^{EM}_{\mu\nu} + F^{GH}_{\mu\nu} = (A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, D_{\mu\nu})$ is self-dual and is also invariant under octonion transformations. The components of split octonion field tensor are the components of split octonion vector field Ψ given by (61). Unified split octonion valued current and split octonion field tensor lead to the unified GDM field equation in the following manner;

$$Q_{\mu\nu,\nu} = J_\mu \quad \Leftrightarrow \quad \overline{\boxdot}\,\boxdot\, V = \begin{pmatrix} J_0 & -\vec{J} \\ \vec{J} & J_0 \end{pmatrix} = J \tag{64}$$

which, on using (56), is equivalent to the following split octonion form of field equation

$$\overline{\boxdot}\Psi = J. \tag{65}$$

Equation (65) is the thus represents the split octonion formulation of GDM field equations for the unified fields of dyons and gravito-dyons.

6 Conclusion

The foregoing analysis describes the combined dynamics of dual invariant unified electromagnetic and gravito-Heavisidian fields with the simultaneous existence of electric, magnetic, gravitational and Heavisidian charges (masses) in compact, simple and consistent way. Though the existence of magnetic and Heavisidian charges is not confirmed, but sound theoretical investigations are in favour of their existence leading to the deeper understanding of fundamental interactions and constituents of matter. From the above analysis it may also be concluded that besides the potential importance of monopoles as intrinsic part of current grand unified theories, monopoles and dyons may provide even more ambitious model to purport the unification of gravitation with strong and electro weak forces. The unified quaternion representation of charges in split octonion basis shows that the dynamics of electric charge is described by the Abelian $U(1)$ gauge structure while the dynamics of other charges have the direct link with $SU(2)$ non Abelian gauge theories leading to their extended structure. Here we have tried to accommodate a new possibility of unification of fundamental interactions in terms of split octonion basis elements where the advanced algebra of octonion is capable to deal the higher dimensional structure of the theory in order to explain the curvature in general relativity at one end and the role of monopoles and dyons in supersymmetry, super-gravity and super strings at the other end. The unified theory presented here hence reproduces the dynamics of electric charge in the absence of other charges. It also reproduces the theory of dyons in the absence of gravito dyons or vice versa. The split octonion formalism may easily be described for the classical and quantum theories fields associated with one, two and four charges while the quantization condition leading to interaction terms describe the combination of dynamics of charges associated with the split octonion basis elements.

Acknowledgements The work is supported by Uttarakhand Council of Science and Technology, Dehradun. One of us OPSN is thankful to Chinese Academy of Sciences and Third world Academy of Sciences for awarding him CAS-TWAS visiting scholar fellowship to pursue a research program in China. He is also grateful to Professor Tianjun Li for his hospitality at Institute of Theoretical Physics, Beijing, China.

References

1. Dirac, P.A.M.: Proc. R. Soc. London A **133**, 60 (1931)
2. Dirac, P.A.M.: Phys. Rev. **74**, 817 (1948)
3. Schwinger, J.: Science **165**, 757 (1978)
4. Schwinger, J.: Phys. Rev. **144**, 1087 (1986)
5. Schwinger, J.: Phys. Rev. **151**, 1048 (1966)
6. Schwinger, J.: Phys. Rev. **151**, 1055 (1966)
7. Schwinger, J.: Phys. Rev. **173**, 1536 (1968)
8. Zwanzinger, D.: Phys. Rev. **176**, 1489 (1968)
9. Zwanzinger, D.: Phys. Rev. D **3**, 880 (1971)
10. Yang, C.N., Wu, T.T.: Nucl. Phys. B **107**, 365 (1976)

11. Yang, C.N., Wu, T.T.: Phys. Rev. D **14**, 437 (1976)
12. Röhrlich, F.: Phys. Rev. **150**, 1104 (1966)
13. Preskill, J.: Ann. Rev. Nucl. Sci. **34**, 461 (1984)
14. Goddard, P., Olive, D.: Rep. Prog. Phys. **41**, 1357 (1978)
15. Craigie, N.: Theory and Detection of Magnetic Monopoles in Gauge Theories. World Scientific, Singapore (1986)
16. Hooft, G.'t: Nucl.Phys. B **79**, 276 (1974)
17. Polyakov, A.M.: JETP Lett. **20**, 194 (1974)
18. Julia, B., Zee, A.: Phys. Rev. D **11**, 2227 (1975)
19. Witten, E.: Phys. Lett. B **86**, 283 (1979)
20. Bisht, P.S., Negi, O.P.S., Rajput, B.S.: IL Nuovo Cimento A **104**, 337 (1991)
21. Bisht, P.S., Negi, O.P.S., Rajput, B.S.: Prog. Theor. Phys. **85**, 157 (1991)
22. Maxwell, J.C.: Philos. Trans. **155**, 492 (1865)
23. Heaviside, O.: Electromagnetic Theory, vol. 1. The Electrician Printing and Publishing Co., London (1894). See Appendix B
24. Heaviside, O.: Electrician **31**, 281–282, 359 (1893)
25. Einstein, A.: Phys. Z. **14**, 1261 (1913)
26. Thirring, H.: Phys. Z. **19**, 204 (1918)
27. Thirring, H.: Phys. Z. **19**, 33 (1918)
28. Thirring, H.: Phys. Z. **22**, 29 (1921)
29. Lense, J., Thirring, H.: Phys. Z. **19**, 156 (1918). The English translation can be found in B. Mashhoon, F.W. Hehl and D.S. Thesis, Gen. Rel. Gravit. **16**, 711 (1984)
30. Ruggiero, M.L., Tartaglia, A.: Gravito magnetic effects. arXiv:gr-qc/0207065v2
31. Dowker, J.S., Roche, J.A.: Proc. R. Soc. **92**, 1 (1967)
32. Singh, A.: Lett. Nuovo Cimento **32**, 232 (1981)
33. Cantani, D.D.: Nuovo Cimento B **60**, 67 (1980)
34. Bisht, P.S., Negi, O.P.S., Rajput, B.S.: Int. J. Theor. Phys. **32**, 2099 (1993)
35. Bisht, P.S., Negi, O.P.S., Rajput, B.S.: Ind. J. Pure Appl. Phys. **28**, 157 (1990)
36. Bisht, P.S., Negi, O.P.S., Rajput, B.S.: Ind. J. Pure Appl. Phys. **32**, 1 (1994)
37. Dickson, L.E.: Ann. Math. **20**, 155 (1919)
38. Hamilton, W.R.: Elements of Quaternions. Chelsea, New York (1969)
39. Tait, P.G.: An Elementary Treatise on Quaternions. Oxford Univ. Press, Oxford (1875)
40. Finklestein, D., Jauch, J.M., Schiminovich, S., Speiser, D.: J. Math. Phys. **4**, 788 (1963)
41. Adler, S.L.: Quaternion Quantum Mechanics and Quantum Fields. Oxford Univ. Press, Oxford (1995)
42. Rajput, B.S., Kumar, S.R., Negi, O.P.S.: Lett. Nuovo **34**, 180 (1982)
43. Rajput, B.S., Kumar, S.R., Negi, O.P.S.: Lett. Nuovo **36**, 75 (1983)
44. Majernik, V.: Physica **39**, 9 (2000)
45. Kravchenkov, V.V.: Applied Quaternion Analysis. Helderman, Berlin (2003)
46. Bisht, S., Bisht, P.S., Negi, O.P.S.: Nuovo Cimento B **113**, 1449 (1998)
47. Morita, K.: Prog. Theor. Phys. **67**, 1860 (1982)
48. Morita, K.: Prog. Theor. Phys. **68**, 2159 (1982). And references cited therein
49. Cayley, A.: Philos. Mag. **26**, 208 (1845)
50. Graves, R.P.: Life of Sir William Rowan Hamilton. Arno Press, New York (1975). 3 volumes
51. Baez, J.C.: The octonions. Bull. Am. Math. Soc. **39**, 145 (2001). math. RA/0105155
52. Catto, S.: Exceptional projective geometries and internal symmetries. hep-th/0302079
53. Tachibana, H., Imaeda, K.: Nuovo Cimento B **104**, 91 (1989)
54. Manogue, C.A., Sudbery, A.: Phys. Rev. D **40**, 4073 (1989)
55. Foot, R., Joshi, G.C.: Int. J. Theor. Phys. **28**, 1449 (1989)
56. Foot, R., Joshi, G.C.: Phys. Lett. B **199**, 203 (1987)
57. Kugo, T., Townsend, P.: Nucl. Phys. B **221**, 357 (1983)
58. Evans, J.M.: Nucl. Phys. B **298**, 92 (1988)
59. Lukierski, J., Toppan, F.: Phys. Lett. B **584**, 315 (2004)
60. Lukierski, J., Toppan, F.: Phys. Lett. B **539**, 266 (2002)
61. Abdel-Khalek Mostafa, K.S.: Ring division algebras, self duality and supersymmetry. arXiv:hep-th/0002155
62. Schray, J.: Octonions and supersymmetry. Ph.D. thesis, Department of Physics, Oregon State University (1994)
63. Duff, M.J. (ed.): The World in Eleven Dimensions: Supergravity, Super Membranes and M-Theory. Institute of Physics Publishing, Bristol (1999). MR 2001b:81002 and references cited therein
64. Gamba, A.: J. Math. Phys. **8**, 775 (1967)
65. Gamba, A.: Il Nuovo Cimento A **111**, 293 (1998)

66. Penny, R.: Am. J. Phys. **36**, 871 (1968)
67. Penny, R.: Il Nuovo Cimento. B **3**, 95 (1971)
68. Imaeda, K., Tachibana, H., Imaeda, M.: Nuovo Cimento B **100**, 53 (1987)
69. Imaeda, K., Tachibana, H., Imaeda, M.: Nuovo Cimento B **105**, 1203 (1990)
70. Gogberashvili, M.: Octonioni version of Dirac equations. hep-th/0505101
71. Gogberashvili, M.: J. Phys. A: Math. Gen. **39**, 7099 (2006)
72. Silagadze, Z.K.: Ann. Fond. Broglie **27**, 241 (2002)
73. Bisht, P.S., Negi, O.P.S.: Ind. J. Pure Appl. Phys. **31**, 292 (1993)
74. Bisht, P.S., Negi, O.P.S.: Ind. J. Pure Appl. Phys. **32**, 183 (1994)
75. Pandey, B., Bisht, P.S., Negi, O.P.S.: Octonion wave equation and generalized fields of dyons. In: Karade, T.M., Khadekar, G.S. (eds.) Proc. National Symp. Mathematical Sciences, vol. 12, p. 137. Nagpur (2001)
76. Dangwal, S., Bisht, P.S., Negi, O.P.S.: Octonionic Gauge formulation for dyonic fields. Had. J. (2007, in press) hep-th/0608061
77. Günaydin, M., Gürsey, F.: Lett. Nuovo Cimento **6**, 407 (1973)
78. Günaydin, M., Gürsey, F.: J. Math. Phys. **14**, 1651 (1973)
79. Günaydin, M., Gürsey, F.: Phys. Rev. D **9**, 3387 (1974)
80. Catto, S., Gürsey, F.: Nuovo Cimento A **86**, 201 (1985)
81. Catto, S., Gürsey, F.: Nuovo Cimento A **99**, 685 (1988)
82. Gürsey, F.: Non-associative algebras in quantum mechanics and particle physics. U. Virginia (1977)
83. Catto, S.: In: Gruber, B. (ed.) Symmetries in Science, vol. 6, p. 129. Plenum, New York (1993)
84. Domokos, G., Domokos Kovesi, S.: Phys. Rev. D **19**, 2984 (1979)
85. Morita, K.: Prog. Theor. Phys. **65**, 787 (1981)
86. Daboul, J., Delbourgo, R.: Matrix representation of octonions and generalizations. hep-th/9906065